

Integrable Structures in Supersymmetric Gauge and String Theory

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Abstract

The effective action of N=2 Yang-Mills theory with adjoint matter is shown to be governed by an integrable spin model with spectral parameter on an elliptic curve. We sketch a route to deriving this effective dynamics from the underlying Yang-Mills theory. Natural generalizations of this structure to all N=2 models, and to string theory, are suggested.

1 Introduction

Recent results in N=2 supersymmetric gauge theory [1, 2, 3, 4] suggest an intimate connection between the low-energy effective theory and integrable systems. In hindsight, this connection is natural given the data that specifies the effective theory: The moduli space of vacua is parametrized by fundamental Casimirs of the Lie algebra; and the BPS mass formula $M(\vec{m}, \vec{n}) = |\vec{m} \cdot \vec{a} + \vec{n} \cdot \vec{a}_D|$ respects a natural $Sp(\text{rank}(G), \mathbf{Z})$ symmetry – where the normalizations a, a_D in this formula may be represented by periods of a special differential λ_{SW} on a Riemann surface $\Sigma[5]$, whose moduli space is that of the gauge theory. On the other hand, precisely the same data characterizes the simplest integrable systems with periodic motions; a set of integrals of motion in one-to-one correspondence with the Casimirs

of a Lie algebra, and evolution that linearizes via the Liouville map on the Jacobian of a Riemann surface Σ (the spectral curve), whose moduli are the integrals of motion. Moreover, the action coordinates are the periods of a special differential pdq around particular cycles (and of course the periods respect a natural symplectic structure). This connection has been made in pure N=2 Yang-Mills theory for all simple gauge groups in [2]. The relevant integrable system is the twisted affine Toda lattice.

This reasoning strongly suggests that one should be able to find an integrable system for any N=2 model. When matter is included, the data for the integrable system must be enlarged to include the masses of matter multiplets, which enter into the algebraic equation for the spectral curve in a particular way[6, 7, 8, 9, 10, 4]. In a recent work, Donagi and Witten[4] showed that, in an ultraviolet finite theory – namely softly broken N=4 Yang-Mills – there is an integrable system with spectral parameter associated to a torus whose modulus is the microscopic gauge coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2}$. Here we will connect these results with those of the author and Warner[2]. Specifically, the integrable system of [4] is in fact the elliptic spin model of Calogero-Moser-Sutherland-Olshanetsky-Perelomov (see for instance [11]). It is known[12] that this system degenerates to the periodic Toda lattice in a certain limit. There is a version of this integrable system for any Lie group[11]; in fact there exists a multiparameter deformation of the potential which is known to preserve integrability[13], and which can degenerate to a variety of trigonometric integrable systems[12] as one degenerates the spectral parameter torus. We will describe this elliptic spin model in section 2. In section 3 we will sketch an idea of how to connect the integrable potential to the underlying Yang-Mills theory. This is followed by a conjecture about how to construct an integrable system for any N=2 theory using superalgebras. In section 4 a natural role of the quantized integrable system is explored. Finally, using recent results of Harvey and Moore[14], we suggest in section 5 that an extension of these ideas to hyperbolic algebras should yield an integrable system governing nonperturbative N=2 string theory.

2 Elliptic spin models

A central point of Donagi and Witten’s work[4] is that finite N=2 theories should have a spectral parameter z living on an elliptic curve. One then

imagines the pure N=2 theory arising from the degeneration of the elliptic curve to a twice-punctured sphere, perhaps while simultaneously shifting some variables. In fact, there is an extremely natural integrable system with this property: The elliptic Calogero-Moser model (the system of type IV in the classification of Olshanetsky and Perelomov[11]). It has as simplest (quadratic) integral of motion

$$H_2 = \frac{1}{2}p^2 + \frac{1}{2} \sum_{\alpha} g^2 \wp(\alpha \cdot q | \tau) \quad (1)$$

where g is a coupling constant (we will soon see it is the mass of the adjoint hypermultiplet in [4]), α are the roots of a Lie algebra \mathfrak{g} , and \wp is the Weierstrass function. There are $\text{rank}(\mathfrak{g})$ canonical pairs of dynamical variables p_i, q_i . The q_i are the vevs of the adjoint Higgs field of the gauge multiplet. Let the periods of the torus be $\omega_1 = 2\pi i$ and ω_2 , with $\tau = -\omega_2/\omega_1$. The reduction to affine Toda is accomplished by the limit[12]

$$\omega_2 \rightarrow \infty \quad , \quad q_j = x_j + c_j \omega_2 \quad , \quad g^2 = e^{b\omega_2} \quad . \quad (2)$$

(at least for the two cases that have been studied in detail in the literature: groups A_n and D_n , where $c_j = (j-1)/h_{\mathfrak{g}}$ and $b = 1/h_{\mathfrak{g}}$; here $h_{\mathfrak{g}}$ is the Coxeter number, which is the same as $h_{\mathfrak{g}}^{\vee}$ for these groups). The dynamics collapses from interactions on the entire root lattice down to exponential interactions for the simple roots (including the affine root) due to the asymptotics (and periodicity) of the Weierstrass function. The limit gives affine Toda with dynamical pair p_i, x_i . The other groups have not yet been checked to see that they reduce properly to affine Toda dynamics on the dual group needed for pure N=2 gauge theory; however, there seem to be enough parameters in the general potentials of [12] to accomodate the Toda theories based on twisted Kac-Moody algebras. One might also generate the twisted affine Toda models using the orbifold constructions of [15]. We do not regard these approaches as particularly natural. Rather, if the promising ideas of section 4 bear fruit, the dual group will arise automatically and in an entirely different way.

Olshanetsky and Perelomov[11] give a Lax pair formulation with Lax operator

$$L = \sum_i p_i \mathbf{h}_i + g \sum_{\alpha} \phi(\alpha \cdot q, z) \mathbf{e}_{\alpha}$$

$$\phi(x, z) = \frac{\sigma(x - z)}{\sigma(x)\sigma(z)} \exp[\zeta(z)x] \quad (3)$$

where σ and ζ are the corresponding Weierstrass functions and z is the spectral parameter. Note that the Lax operator has very much the structure required by Donagi and Witten: holomorphic except at $z = 0$ where it has a simple pole; the residue is the operator $\sum_{\alpha} \mathbf{e}_{\alpha}$ which for A_n has eigenvalues $(1, 1, \dots, 1, -(n-1))$ when written as a matrix in the fundamental representation¹. The overall coefficient of this residue was argued in [4] to be the mass μ of the adjoint hypermultiplet; hence we identify g as that mass. This system fits rather nicely into the Hitchin framework as well, *c.f.* [16, 17]. In fact, Nekrasov[16] and Olshanetsky[17] have shown that the classical Hitchin system on an elliptic curve with bundle monodromy $\text{diag}(\exp[2\pi i q \cdot \mathbf{h}])$ generates a (Hitchin system, not Yang-Mills) Higgs vev which is essentially (3), when one chooses an appropriate coadjoint orbit at the pole (related to a certain symmetric space[11, 18, 19]).

3 A route to integrability?

The finite $N=2$ models are special in that one still has control over the microscopic gauge coupling in the effective theory; the action is not renormalized. Thus one might imagine deriving the elliptic spin model by considering the dynamics of the low-energy degrees of freedom. In particular, much of the structure of the low-energy theory seems to be governed by the fermion zero-modes. For instance, in asymptotically free theories they determine the number of vacuum states, the discrete breaking of R -symmetry, and so on. In the analogous two-dimensional $N=2$ models, an important role is played by the fermion zero-mode Hilbert space as a function of the parameters[21, 22]. Thus it is intriguing that terms can be generated in the effective potential for the fermions which have very much the structure of the potential in (1). Consider integrating out the massive vector multiplets of the root generators of \mathfrak{g} in the Yang-Mills action. Their masses (*e.g.* think of the scalar component) come from couplings like $\text{tr}\{[\phi^{\dagger}, \phi]^2\}$; the mass of ϕ_{α} is roughly $(\alpha \cdot v)^2$, where $v = \langle \phi \rangle$ (it is perhaps better to take v real for the moment). Now let

¹I thank E. Witten for pointing out the relation between the residue of (3) and the section of the Higgs bundle in [4].

us consider the effect on the low-energy fermion modes of integrating out the massive vector multiplets. This is induced by the Yukawa coupling $\phi_\alpha \bar{\psi} t_{-\alpha} \psi$. If we call $\bar{\psi} t_\alpha \psi = S_\alpha$, integrating out ϕ_α induces a potential

$$\sum_{\alpha} \frac{S_{\alpha} S_{-\alpha}}{(\alpha \cdot v)^2} \quad (4)$$

This should be accurate for weak Yang-Mills coupling and small Higgs vev v . However we know how to extend this to a more precise answer. Recall the work of Harvey *et.al.* [20] on the dimensional reduction of super-Yang-Mills: If we consider the $N = 4$ theory to come from higher dimensions, compactified on a torus, then $\alpha \cdot v$ should be periodically identified due to gauge transformations in the higher-dimensional theory. Then the potential (4) is that of the spin generalization of the trigonometric Calogero-Moser-Sutherland system (*c.f.* [16, 19])! That is, if the spin state is one with symmetry $S_{\alpha} = \text{const.}$ independent of α , then one obtains the ordinary Calogero-Moser-Sutherland potential $(1/\sin^2)$. Finally, if we assume that the spectral parameter of our system lives on a torus, the above potential should be further generalized to a doubly-periodic function. Perhaps this periodic structure will come from a better understanding of the field space of the adjoint Higgs away from the origin. In any event, the natural setting seems to include a set of fermions living on the spectral parameter torus. That is, one imagines a fiber bundle with fiber the Hilbert space of fermion zero modes, and base space the total space of the universal elliptic curve (parametrized by the spectral variable z) over the moduli space of Higgs vevs v and gauge couplings τ ; one would then try to perform an analysis along the lines of [21, 22]².

We now see a possible route to getting all finite N=2 theories: Consider N=2 Yang-Mills coupled to an appropriate combination of hypermultiplets in representations R_i of G , yielding a finite theory. Integrating out the massive vector multiplets will yield a potential exactly as above, except that S_{α} has a sum of contributions from the generator $S_{\alpha}(R_i)$ of each hypermultiplet. One thing that needs to be explained is how vector multiplet fermions and hypermultiplet fermions enter with opposite signs in situations like the perturbative beta function $\beta_{\tau} \propto [2C_2(G) - \sum_i T_2(R_i)]$. One might then understand the finiteness of the spacetime N=2 theory as the vanishing of a Chern class or holomorphic anomaly of fermions living on the spectral parameter

²This line of reasoning was proposed already in [2].

curve; for instance, the fermion currents S_α have a current algebra anomaly $T_2(R_i)$ (equals $C_2(G)$ for the adjoint representation) if they are thought of as two-dimensional fermions on the spectral parameter curve.

A somewhat different approach is suggested by the string theory work of Harvey and Moore[14]. There, a superalgebra was defined on BPS states, with the even generators corresponding to vector multiplets, and the odd generators corresponding to hypermultiplets. This indicates that one might consider an integrable system based on a Lie superalgebra for the N=2 theories with matter. This is particularly natural if one follows the logic of Seiberg[23] and regards the hypermultiplet masses as arising from the expectation value of a field. Indeed, one may gauge the flavor group³ but look in the weak-coupling limit ($e_{flavor} \rightarrow 0$ for finite theories, $\Lambda_{flavor} \rightarrow 0$ for asymptotically free ones); then the N_f hypermultiplet masses are the flavor group Higgs vevs, but the flavor dynamics is decoupled (the corresponding spectral curve is completely degenerate, so only poles will arise in the integral that normalizes the part of the BPS mass formula coming from hypermultiplet masses). Of course, one need not specialize to this limit; in general there will be an interesting interplay between the spectral covers for the two groups. In the end, one is looking for superalgebras whose bosonic part is $G_c \times G_f$ and whose odd part transforms in the proper representations of each group. Thus for $SU(N_c)$ with N_f fundamental flavors, one should consider the supergroup $SU(N_c|N_f)$; for $SO(N_c)$ gauge theory, $OSp(N_c|N_f)$; and for $Sp(N_c)$, gauge theory $OSp(2N_f|N_c)$. Higgs branches of the moduli space of vacua of the N=2 gauge theory should be realized when one puts nontrivial holonomy (of the Hitchin Higgs bundle over the spectral parameter torus) in the odd directions of the group, corresponding to giving a vacuum expectation value to a hypermultiplet. Similarly, the adjoint hypermultiplet of the softly broken N=4 theory ought to be a fermionic component of the Hitchin system of the previous section. A logical candidate is the supersymmetric version of Hitchin's construction, with the bosonic fields related to the adjoint vector multiplet and their superpartners to the adjoint hypermultiplet. It is not immediately clear what this ansatz has to do with the above analysis of the low-energy fermion dynamics.

Thus the use of superalgebras opens the door to a unified description of the Higgs and abelian Coulomb phases. It is hoped that a combination of

³I thank N. Warner for discussions about gauging flavor symmetry in N=2 theories.

the ideas presented here and those of [5, 6, 4] will enable a unified description of all phases of supersymmetric gauge theory. Control over the microscopic couplings provided by finiteness may also lead to a proof of N=1 dualities[24].

4 Quantization of the spin model

One of the key observations of [1, 2, 3] is that the effective prepotential of pure N=2 Yang-Mills gauge theory is the free energy of the *Whitham averaged* hierarchy of the underlying integrable system. The averaging procedure is an adiabatic deformation dynamics on the moduli space of the original integrable system (in this case twisted affine Toda). An alternate perspective on this procedure is that it is the first step in the WKB quantization of the integrable system[25]. This suggests that, to some degree, we should not be considering the classical integrable system but rather its quantum counterpart. The quantization of the spin generalization of the Hamiltonian (1) has an extremely interesting interpretation: It is essentially the RHS of the Knizhnik-Zamolodchikov-Bernard equation for the conformal blocks of the WZW theory on the once-punctured torus. More precisely, the KZB equation in this case is[26]

$$4\pi i(k + h_{\mathbf{g}}^{\vee}) \frac{\partial}{\partial \tau} \tilde{\omega}(z, \tau, \vec{q}) = \frac{\partial^2}{\partial \vec{q}^2} \tilde{\omega} - \eta_1(\tau) C_2(V) \tilde{\omega} - \sum_{\alpha \in \Delta} \wp(\vec{\alpha} \cdot \vec{q}) \mathbf{e}_{\alpha} \mathbf{e}_{-\alpha} \tilde{\omega} \quad (5)$$

where $\tilde{\omega}(z, \tau, \vec{q}) = \Pi(\tau, \vec{q}) \omega(z, \tau, \vec{q})$ with Π the Weyl-Kac denominator, ω a conformal block of the once-punctured torus with representation V at the puncture; $\eta_1(\tau)$ is the first nonpole term in the expansion of $\partial_z \log[\vartheta_1(z|\tau)]$ near $z = 0$. At the critical level $k = -h_{\mathbf{g}}^{\vee}$, the KZB equation coincides with the action of the quadratic quantum Hamiltonian of the Hitchin system on the quantum Hilbert space[16, 17]. In this case, the Hamiltonian (and other integrals of motion) falls into the center $\mathcal{C}_{-h_{\mathbf{g}}^{\vee}}$ of the universal enveloping algebra $U_k(L\mathbf{g})$ at $k = -h_{\mathbf{g}}^{\vee}$ [27, 28, 29]. The analogue of this center when $k \neq h_{\mathbf{g}}^{\vee}$ is the W -algebra $W_k(\mathbf{g})$ [28]. There is a remarkable duality $W_k(\mathbf{g}) \approx W_{k^{\vee}}(\mathbf{g}^{\vee})$ [28], related to the geometric Langlands program [27]. Here $r^{\vee}(k + h_{\mathbf{g}}^{\vee}) = (k^{\vee} + h_{\mathbf{g}^{\vee}}^{\vee})^{-1}$, with r^{\vee} the maximal number of edges connecting two vertices of the Dynkin diagram of \mathbf{g} (*i.e.* the order of the diagram automorphism in the orbifold construction of twisted Kac-Moody algebras). Thus it would seem that the WZW model near the critical level (related

to the quantized Hitchin system), is related to the WKB limit $k \rightarrow \infty$ of the dual Kac-Moody; which may explain the appearance of WKB averaged, *twisted* affine Toda in the pure gauge N=2 theory[2] if one can prove that the partition function of the finite theory satisfies the KZB equation in the critical level limit. This duality is also natural when one considers that the ultraviolet limit of the Yang-Mills theory is N=4 with gauge group G (and integrable system elliptic Calogero-Moser on G), whereas the infrared theory is an N=2 gauge theory of monopoles with gauge group G^\vee (and integrable system affine Toda on $(LG)^\vee$). The transition between the two descriptions should be made natural by the renormalization group flow corresponding to the range of theories with adjoint masses between $\mu = 0$ and $\mu = \infty$. It would be amusing if, as appears to be the case, Montonen-Olive and Langlands duality were one and the same.

5 Extension to string theory

Recent work of Harvey and Moore[14] indicates that the above ideas have an elegant and extremely natural generalization to string theory. These authors found that the structure of threshold corrections in the N=2 Heterotic string (*e.g.* compactified on $K3 \times T^2$) were organized by the structure of a generalized or hyperbolic Kac-Moody (super)algebra. At first sight one might doubt that the above setting could be carried over to string theory, because the beta function of the theory does not vanish. Nevertheless, the theory is finite (because it's string theory), and one might regard the breaking of supersymmetries by the compactification manifold to be a soft breaking (it is for instance restored in the large radius limit)⁴. The complex coupling constant of the theory is the expectation value S of the dilaton multiplet. Thus we should look for an integrable system on a spectral parameter torus with modulus $\tau = S$. The gauge group has been identified by Harvey and Moore[14]: a hyperbolic Kac-Moody (super)algebra acting on the BPS states. The proof of integrability of models like (1),(3) relies on little more than that $\mathbf{e}_\alpha, \mathbf{h}_i$ are the generators of a Lie algebra, and so should hold for the hyperbolic case. The dynamical coordinates of the integrable system will be the Higgs vevs \vec{v} in the CSA (expectation values of Wilson lines in [14]), together

⁴Also, even in nonsupersymmetric solutions the spectrum looks asymptotically supersymmetric[30].

with the expectation values T, U of the Narain moduli corresponding to the additional Cartan generators of the hyperbolic algebra. Thus we conjecture that (3) will be the Lax operator of the integrable system governing nonperturbative N=2 Heterotic string theory, with $\vec{q} = (T, U, \vec{v})$, $\tau = S$, and \mathfrak{g} the string gauge (super)algebra. Using the odd generators of the superalgebra, again one might reduce the rank of the low-energy gauge group by passing to a Higgs branch along the lines suggested in section 3.

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